Effective Levi-Civita Dilaton theory from Metric Affine Dilaton Gravity

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Abstract

We show how a Metric Affine theory of Dilaton gravity can be reduced to an effective Riemannian Dilaton gravity model. A simple generalization of the Obukhov-Tucker-Wang theorem to Dilaton gravity is then presented.

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Among the four fundamental interactions, the two feeble are characterised by dimensional coupling constants, $G_F = (300Gev)^{-2}$ and Newton's coupling constant $G_N = (10^{19}Gev)^{-2}$.
It is well known that interactions with dimensional coupling constants present many problems among which there is the renormalizability.
The success of the Weinberg-Salam model has told us that the weak interaction is characterised by a dimensionless coupling constant and the dimensions of G_F are due to the spontaneous symmetry breaking mechanism, so that $G_F \cong \frac{1}{v_W{}^2}$ where $v_W \cong 300Gev$ is the vacuum expectation value of the Higgs field.
The weakness of the weak interaction being related to the large vacuum expectation value of the scalar field [1].
It is believed that similar mechanisms may occur for gravity, which is characterised by a dimensionless coupling constant ξ. The weakness of gravity then would be related to the symmetry breaking at very high energies [2-4].
This is obtained starting from a Dilaton theory which presents Weyl scale invariance. The potential $V(\psi)$ which appears in the action is assumed to have its minimum at $\psi = \sigma$, then when $\psi = \sigma$ the Dilaton theory reduces to the Einstein-Hilbert action with gravitational constant $G_N = \frac{1}{8\pi\xi\sigma^2}$.
In this Letter we investigate in the Tucker-Wang approach to non Riemannian gravity the action:

$$S = \int k\psi^2 R \star 1 + \beta(d\psi \wedge \star d\psi) - V(\psi) \star 1 \tag{1}$$

Where R is the scalar curvature associated with the full non Riemannian connection.
In the Tucker-Wang approach to MAG we choose the metric to be orthonormal $g_{ab} = \eta_{ab} = (-1, 1, 1, 1, ...)$ and we vary with respect to the coframe e^a and the connection $\omega^a{}_b$ considered as independent gauge potentials.
As we will see the non Riemannian contribution to the Einstein-Hilbert term times ψ^2 is equivalent in the field equations to a kinetic term for the Dilaton and if no torsion terms are explicitly introduced in the action, the coupling ξ is not arbitrary contrary to what happens in Ref [5-7] where ξ is a free parameter.
Once β is fixed we obtain an effective ξ'^{-1} given by $\xi^{-1} + 4\frac{n-1}{n-2}$ (see eq. 22).
It has to be observed however that since we have the condition $g_{ab} = \eta_{ab}$, in general the Weyl group for the action (1) is not defined in the usual way

but we may still introduce a Weyl rescaling of the form $e^a \to f\, e^a$ for the coframe, with f an arbitrary function of the spacetime..
The variation of (1) with respect to ψ gives:

$$-\beta d \star d\psi + k\psi(R \star 1) - V'(\psi) \star 1 = 0 \tag{2}$$

By considering the variation with respect to the connection we get the equation:

$$D \star (e_a \wedge e^b) = -\frac{2}{\psi}[d\psi \wedge \star(e_a \wedge e^b)] = A(\psi)[d\psi \wedge \star(e_a \wedge e^b)] \tag{3}$$

with $A(\psi) = -\frac{2}{\psi}$.
To prove the previous one we have to observe that the Einstein-Hilbert term which appears in the action (1) can be written as:

$$R \star 1 = R^a{}_b \wedge \star(e_a \wedge e^b) \tag{4}$$

Where $R^a{}_b$ are the curvature two forms which are defined by $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$. So we get:

$$R \star 1 = (d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b) \wedge \star(e_a \wedge e^b) \tag{5}$$

Then we have to calculate the variation of:

$$\psi^2\,(d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b) \wedge \star(e_a \wedge e^b) \tag{6}$$

We have:

$$\begin{aligned} \psi^2\,(d\omega^a{}_b \wedge \star(e_a \wedge e^b)) = \\ d[\psi^2\,\omega^a{}_b \wedge \star(e_a \wedge e^b)] + \omega^a{}_b \wedge d[\psi^2\, \star(e_a \wedge e^b)] \end{aligned} \tag{7}$$

so (mod d):

$$\begin{aligned} d\omega^a{}_b \wedge \star(e_a \wedge e^b)\psi^2 = \\ \omega^a{}_b \wedge d[\psi^2\, \star(e_a \wedge e^b)] = \\ \omega^a{}_b \wedge \psi^2\, d(\star(e_a \wedge e^b)) + 2\,\psi(\omega^a{}_b \wedge d\psi \wedge \star(e_a \wedge e^b)) \end{aligned} \tag{8}$$

Considering the connection variation and using the definition of the covariant exterior derivative D [8] we obtain formula (3).
The full non Riemannian Einstein Hilbert term can be written as:

$$R \star 1 = \overset{o}{R} \star 1 - \hat{\lambda}^a{}_c \wedge \hat{\lambda}^c{}_b \wedge \star(e^b \wedge e_a) - d(\hat{\lambda}^a{}_b \wedge \star(e^b \wedge e_a)) \tag{9}$$

where $\hat{\lambda}^a{}_b$ is the traceless part of the non Riemannian part of the connection $\lambda^a{}_b$.
By considering the coframe variation we get then the generalized Einstein equations:

$$k\psi^2 \overset{o}{R}{}^a{}_b \wedge \star(e_a \wedge e^b \wedge e_c) - 2k\psi[\hat{\lambda}^a{}_b \wedge d\psi \wedge \star(e^b \wedge e_a \wedge e_c)] - \beta[d\psi \wedge i_c \star d\psi + i_c d\psi \wedge \star d\psi] + k\psi^2[\hat{\lambda}^a{}_d \wedge \hat{\lambda}^d{}_b] \wedge \star(e_a \wedge e^b \wedge e_c) - V(\psi) \star e_c = 0 \tag{10}$$

The Cartan equation can be written as:

$$D \star (e^a \wedge e_b) = A(\psi)[d\psi \wedge \star(e^a \wedge e_b)] = F^a{}_b \tag{11}$$

To solve the previous we need the $0-forms$ $f^{ca}{}_b$ defined by $F^a{}_b = f^{ca}{}_b \star e_c$.
We decompose the nonmetricity and torsion as:

$$\begin{aligned} Q_{ab} &= \hat{Q}_{ab} + \frac{1}{n} g_{ab} Q \\ T_a &= \hat{T}_a + \frac{1}{n-1}(e_a \wedge T) \end{aligned} \tag{12}$$

where $Q_{ab} = Dg_{ab}, T_a = de_a + \omega_a{}^b \wedge e_b, Q = Q^a{}_a, T = i_a T^a$.
We have the relations:

$$\begin{aligned} \hat{Q_{bc}} &= \frac{1}{n} g_{bc}(f^d{}_{da} + f^d{}_{ad})e^a - (f_{bac} + f_{bca} - f_{abc})e^a \\ \hat{T}_c &= \frac{1}{n-1}(e_c \wedge e^a) f^d{}_{ad} - \frac{1}{2}(e^b \wedge e^a)(f_{bac} + f_{bca} + f_{cab}) \\ T - \frac{n-1}{2n} Q &= \frac{1}{n(n-2)}(f^c{}_{ac} + (1-n) f^c{}_{ca})e^a \end{aligned} \tag{13}$$

We get:

$$f_{cab} = A(\psi) i_c(\star(d\psi \wedge \star(e_a \wedge e_b))) \tag{14}$$

from which we get:

$$\begin{aligned} \hat{Q}^{ab} &= 0 \\ \hat{T}_c &= 0 \end{aligned} \tag{15}$$

and

$$T = \frac{n-1}{2n} Q + \frac{1-n}{n-2} A(\psi) d\psi \tag{16}$$

the solution for the nonmetricity and torsion can then be written as:

$$\begin{aligned} Q_{ab} &= \frac{1}{n} g_{ab} Q \\ T^a = \frac{1}{2n}(e^a \wedge Q) &- \frac{1}{n-2}(e^a \wedge d\psi) A(\psi) \end{aligned} \tag{17}$$

Using the expression of $\lambda^a{}_b$ as a function of T^a and Q_{ab}:

$$2\lambda_{ab} = i_a T_b - i_b T_a - (i_a i_b T_c + i_b Q_{ac} - i_a Q_{bc}) e^c - Q_{ab} \tag{18}$$

we get:

$$\lambda_{ab} = -\frac{1}{2n} g_{ab} Q + \frac{1}{n-2} A(\psi)(i_a(d\psi) e_b - i_b(d\psi) e_a) \tag{19}$$

and the traceless part:

$$\hat{\lambda}_{ab} = \frac{1}{n-2} A(\psi)(i_a(d\psi) e_b - i_b(d\psi) e_a) \tag{20}$$

By using the previous expression in the generalised Einstein equations we get after some calculations:

$$k\psi^2 \overset{o}{G}_c - \beta'[d\psi \wedge i_c \star d\psi + i_c d\psi \wedge \star d\psi] - V(\psi) \star e_c = 0 \tag{21}$$

where $\overset{o}{G}_c = \overset{o}{R}{}^a{}_b \wedge \star(e_a \wedge e^b \wedge e_c)$ and:

$$\beta' = \beta + 4k \frac{n-1}{n-2} \tag{22}$$

Then if we choose:

$$\beta = -4k \frac{n-1}{n-2} \tag{23}$$

we get the generalized Einstein equations reduced to:

$$k\psi^2 \overset{o}{G}_c - V(\psi) \star e_c = 0 \tag{24}$$

which are equivalent to:

$$k\overset{o}{G}_c - V(\psi)\psi^{-2} \star e_c = 0 \tag{25}$$

This are the Einstein equations we would have obtained from an Einstein theory with the potential $V(\psi)\psi^{-2}$.
The non Riemannian contribution to the Einstein-Hilbert term is:

$$\Delta R \star 1 = -\frac{4}{\psi^2}\frac{n-1}{n-2}(d\psi \wedge \star d\psi) \tag{26}$$

so, the equation for ψ becomes:

$$-\beta d \star d\psi + k\psi(\overset{o}{R} \star 1) - k\frac{n-1}{n-2}\frac{4}{\psi}(d\psi \wedge \star d\psi) - V'(\psi) \star 1 = 0 \tag{27}$$

Observe the following interesting case.
Suppose we start from the $\beta = 0$ in the action (1) that is:

$$S = \int k\psi^2 R \star 1 - V(\psi) \star 1 \tag{28}$$

then we get the equations:

$$\begin{aligned} k\overset{o}{G}_c\psi^2 - 4k\frac{n-1}{n-2}[d\psi \wedge i_c \star d\psi + i_c d\psi \wedge \star d\psi] - V(\psi) \star e_c = 0 \\ +k\psi \overset{o}{R} \star 1 - \frac{4}{\psi}\frac{n-1}{n-2}(d\psi \wedge \star d\psi) - V'(\psi) \star 1 = 0 \end{aligned} \tag{29}$$

For $n = 4$ we get:

$$\begin{aligned} k\overset{o}{G}_c\psi^2 - 6k[d\psi \wedge i_c \star d\psi + i_c d\psi \wedge \star d\psi] - V(\psi) \star e_c = 0 \\ +k\psi \overset{o}{R} \star 1 - \frac{6}{\psi}(d\psi \wedge \star d\psi) - V'(\psi) \star 1 = 0 \end{aligned} \tag{30}$$

The Einstein equations in this case coincide formally with the conformally invariant Einstein equations obtained starting from the action:

$$S = \int k\psi^2 \overset{o}{R} \star 1 + 6k(d\psi \wedge \star d\psi) - V(\psi) \star 1 \tag{31}$$

We have to remember however that this equivalence holds with the amendment that the Weyl rescaling is defined for the coframe and not for the metric since g_{ab} is fixed to be orthonormal.
What found above can be extended to more general actions like for example:

$$S = \int k\psi^2 R\star 1+\beta(d\psi\wedge\star d\psi)+f_1(\psi)\frac{\alpha}{2}(dQ\wedge\star dQ)+f_2(\psi)\frac{\gamma}{2}(Q\wedge\star Q)-V(\psi)\star 1 \tag{32}$$

where Q is the trace of the nonmetricity 1-forms $Q = g_{ab}Q^{ab}$ and $f_1(\psi), f_2(\psi)$ are two arbitrary functions of the Dilaton field ψ.
We get in this case the generalized Einstein equations:

$$\begin{aligned} k\psi^2 \overset{o}{G}_c - \beta'[d\psi\wedge i_c \star d\psi + i_c d\psi\wedge \star d\psi] - V(\psi)\star e_c + \quad (33) \\ -f_1(\psi)\frac{\alpha}{2}[dQ\wedge i_c\star dQ + i_c dQ\wedge\star dQ] + f_2(\psi)\frac{\gamma}{2}[Q\wedge i_c\star Q - i_c Q\wedge\star Q] = 0 \end{aligned}$$

we get the equation for Q.

$$\alpha d(f_1(\psi)\star dQ) + f_2(\psi)\gamma\star Q = 0 \tag{34}$$

Eq. (33) can be considered as a generalization of the Obukhov-Tucker-Wang theorem [9-11] to the Dilaton Gravity action (32).
The equation for ψ (27) contains other two terms which are:

$$\frac{\alpha}{2}f_1'(\psi)(dQ\wedge\star dQ) + \frac{\gamma}{2}f_2'(\psi)(Q\wedge\star Q) \tag{35}$$

If $\beta = -4k\frac{n-1}{n-2}$ then the Einstein equations would reduce to:

$$\begin{aligned} k\psi^2\overset{o}{G}_c - \quad (36) \\ V(\psi)\star e_c - f_1(\psi)\frac{\alpha}{2}[dQ\wedge i_c\star dQ + i_c dQ\wedge\star dQ] \\ +f_2(\psi)\frac{\gamma}{2}[Q\wedge i_c\star Q - i_c Q\wedge\star Q] = 0 \end{aligned}$$

The inclusion of torsion terms in the action like $T\wedge\star T$ and $T^c\wedge\star T_c$ would

complicate the analysis since in that case the traceless part of the Cartan equation and then the expression for $\lambda^a{}_b$ would be modified [10], the study

of these more general cases as well as the study of further possible generalizations of Obukhov-Tucker-Wang theorem will be considered in following investigations.

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